PEARSON NEW INTERNATIONAL EDITION

Linear Algebra

S. Friedberg A. Insel L. Spence

Fourth Edition

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Table of Contents

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1

Vector Spaces

1.1 INTRODUCTION

Many familiar physical notions, such as forces, velocities,[1] and accelerations, involve both a magnitude (the amount of the force, velocity, or acceleration) and a direction. Any such entity involving both magnitude and direction is called a "vector." A vector is represented by an arrow whose length denotes the magnitude of the vector and whose direction represents the direction of the vector. In most physical situations involving vectors, only the magnitude and direction of the vector are significant; consequently, we regard vectors with the same magnitude and direction as being equal irrespective of their positions. In this section the geometry of vectors is discussed. This geometry is derived from physical experiments that test the manner in which two vectors interact.

Familiar situations suggest that when two like physical quantities act simultaneously at a point, the magnitude of their effect need not equal the sum of the magnitudes of the original quantities. For example, a swimmer swimming upstream at the rate of 2 miles per hour against a current of 1 mile per hour does not progress at the rate of 3 miles per hour. For in this instance the motions of the swimmer and current oppose each other, and the rate of progress of the swimmer is only 1 mile per hour upstream. If, however, the

[1]The word *velocity* is being used here in its scientific sense—as an entity having both magnitude and direction. The magnitude of a velocity (without regard for the direction of motion) is called its **speed**.

swimmer is moving downstream (with the current), then his or her rate of progress is 3 miles per hour downstream.

Experiments show that if two like quantities act together, their effect is predictable. In this case, the vectors used to represent these quantities can be combined to form a resultant vector that represents the combined effects of the original quantities. This resultant vector is called the *sum* of the original vectors, and the rule for their combination is called the *parallelogram law.* (See Figure 1.1.)

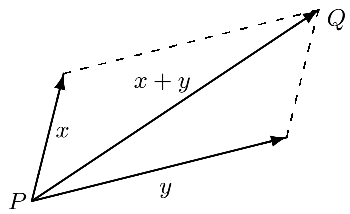

Figure 1.1

Parallelogram Law for Vector Addition. *The sum of two vectors x and y that act at the same point P is the vector beginning at P that is represented by the diagonal of parallelogram having x and y as adjacent sides.*

Since opposite sides of a parallelogram are parallel and of equal length, the endpoint Q of the arrow representing $x + y$ can also be obtained by allowing x to act at P and then allowing y to act at the endpoint of x. Similarly, the endpoint of the vector $x + y$ can be obtained by first permitting y to act at P and then allowing x to act at the endpoint of y. Thus two vectors x and y that both act at the point P may be added "tail-to-head"; that is, either x or y may be applied at P and a vector having the same magnitude and direction as the other may be applied to the endpoint of the first. If this is done, the endpoint of the second vector is the endpoint of $x + y$.

The addition of vectors can be described algebraically with the use of analytic geometry. In the plane containing x and y, introduce a coordinate system with P at the origin. Let (a_1, a_2) denote the endpoint of x and (b_1, b_2) denote the endpoint of y. Then as Figure 1.2(a) shows, the endpoint Q of $x+y$ is $(a_1 + b_1, a_2 + b_2)$. Henceforth, when a reference is made to the coordinates of the endpoint of a vector, the vector should be assumed to emanate from the origin. Moreover, since a vector beginning at the origin is completely determined by its endpoint, we sometimes refer to *the point x* rather than *the endpoint of the vector x* if x is a vector emanating from the origin.

Besides the operation of vector addition, there is another natural operation that can be performed on vectors—the length of a vector may be magnified

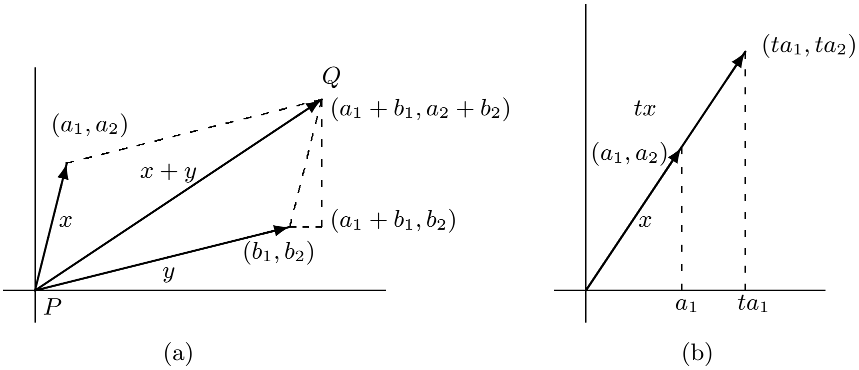

Figure 1.2

or contracted. This operation, called *scalar multiplication*, consists of multiplying the vector by a real number. If the vector x is represented by an arrow, then for any real number t, the vector tx is represented by an arrow in the same direction if $t \geq 0$ and in the opposite direction if $t < 0$. The length of the arrow tx is $|t|$ times the length of the arrow x. Two nonzero vectors x and y are called **parallel** if $y = tx$ for some nonzero real number t. (Thus nonzero vectors having the same or opposite directions are parallel.)

To describe scalar multiplication algebraically, again introduce a coordinate system into a plane containing the vector x so that x emanates from the origin. If the endpoint of x has coordinates (a_1, a_2), then the coordinates of the endpoint of tx are easily seen to be (ta_1, ta_2). (See Figure 1.2(b).)

The algebraic descriptions of vector addition and scalar multiplication for vectors in a plane yield the following properties:

1. For all vectors x and y, $x + y = y + x$.
2. For all vectors x, y, and z, $(x + y) + z = x + (y + z)$.
3. There exists a vector denoted $\mathit{0}$ such that $x + \mathit{0} = x$ for each vector x.
4. For each vector x, there is a vector y such that $x + y = \mathit{0}$.
5. For each vector x, $1x = x$.
6. For each pair of real numbers a and b and each vector x, $(ab)x = a(bx)$.
7. For each real number a and each pair of vectors x and y, $a(x + y) = ax + ay$.
8. For each pair of real numbers a and b and each vector x, $(a + b)x = ax + bx$.

Arguments similar to the preceding ones show that these eight properties, as well as the geometric interpretations of vector addition and scalar multiplication, are true also for vectors acting in space rather than in a plane. These results can be used to write equations of lines and planes in space.

Consider first the equation of a line in space that passes through two distinct points A and B. Let O denote the origin of a coordinate system in space, and let u and v denote the vectors that begin at O and end at A and B, respectively. If w denotes the vector beginning at A and ending at B, then "tail-to-head" addition shows that $u+w=v$, and hence $w=v-u$, where $-u$ denotes the vector $(-1)u$. (See Figure 1.3, in which the quadrilateral $OABC$ is a parallelogram.) Since a scalar multiple of w is parallel to w but possibly of a different length than w, any point on the line joining A and B may be obtained as the endpoint of a vector that begins at A and has the form tw for some real number t. Conversely, the endpoint of every vector of the form tw that begins at A lies on the line joining A and B. Thus an equation of the line through A and B is $x = u + tw = u + t(v-u)$, where t is a real number and x denotes an arbitrary point on the line. Notice also that the endpoint C of the vector $v-u$ in Figure 1.3 has coordinates equal to the difference of the coordinates of B and A.

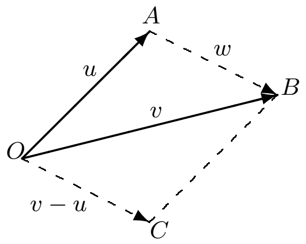

Figure 1.3

Example 1

Let A and B be points having coordinates $(-2,0,1)$ and $(4,5,3)$, respectively. The endpoint C of the vector emanating from the origin and having the same direction as the vector beginning at A and terminating at B has coordinates $(4,5,3)-(-2,0,1)=(6,5,2)$. Hence the equation of the line through A and B is

$$x = (-2,0,1) + t(6,5,2). \qquad \blacklozenge$$

Now let A, B, and C denote any three noncollinear points in space. These points determine a unique plane, and its equation can be found by use of our previous observations about vectors. Let u and v denote vectors beginning at A and ending at B and C, respectively. Observe that any point in the plane containing A, B, and C is the endpoint S of a vector x beginning at A and having the form $su+tv$ for some real numbers s and t. The endpoint of su is the point of intersection of the line through A and B with the line through S

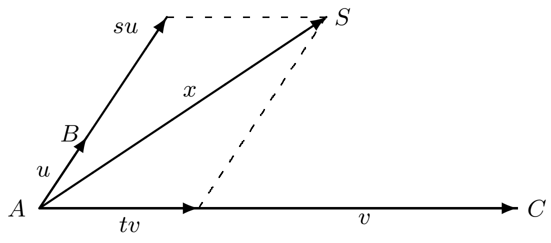

Figure 1.4

parallel to the line through A and C. (See Figure 1.4.) A similar procedure locates the endpoint of tv. Moreover, for any real numbers s and t, the vector $su + tv$ lies in the plane containing A, B, and C. It follows that an equation of the plane containing A, B, and C is

$$x = A + su + tv,$$

where s and t are arbitrary real numbers and x denotes an arbitrary point in the plane.

Example 2

Let A, B, and C be the points having coordinates $(1, 0, 2)$, $(-3, -2, 4)$, and $(1, 8, -5)$, respectively. The endpoint of the vector emanating from the origin and having the same length and direction as the vector beginning at A and terminating at B is

$$(-3, -2, 4) - (1, 0, 2) = (-4, -2, 2).$$

Similarly, the endpoint of a vector emanating from the origin and having the same length and direction as the vector beginning at A and terminating at C is $(1, 8, -5) - (1, 0, 2) = (0, 8, -7)$. Hence the equation of the plane containing the three given points is

$$x = (1, 0, 2) + s(-4, -2, 2) + t(0, 8, -7). \quad \blacklozenge$$

Any mathematical structure possessing the eight properties on page 3 is called a *vector space*. In the next section we formally define a vector space and consider many examples of vector spaces other than the ones mentioned above.

EXERCISES

1. Determine whether the vectors emanating from the origin and terminating at the following pairs of points are parallel.

(a) $(3, 1, 2)$ and $(6, 4, 2)$
(b) $(-3, 1, 7)$ and $(9, -3, -21)$
(c) $(5, -6, 7)$ and $(-5, 6, -7)$
(d) $(2, 0, -5)$ and $(5, 0, -2)$

2. Find the equations of the lines through the following pairs of points in space.

(a) $(3, -2, 4)$ and $(-5, 7, 1)$
(b) $(2, 4, 0)$ and $(-3, -6, 0)$
(c) $(3, 7, 2)$ and $(3, 7, -8)$
(d) $(-2, -1, 5)$ and $(3, 9, 7)$

3. Find the equations of the planes containing the following points in space.

(a) $(2, -5, -1)$, $(0, 4, 6)$, and $(-3, 7, 1)$
(b) $(3, -6, 7)$, $(-2, 0, -4)$, and $(5, -9, -2)$
(c) $(-8, 2, 0)$, $(1, 3, 0)$, and $(6, -5, 0)$
(d) $(1, 1, 1)$, $(5, 5, 5)$, and $(-6, 4, 2)$

4. What are the coordinates of the vector *0* in the Euclidean plane that satisfies property 3 on page 3? Justify your answer.

5. Prove that if the vector x emanates from the origin of the Euclidean plane and terminates at the point with coordinates (a_1, a_2), then the vector tx that emanates from the origin terminates at the point with coordinates (ta_1, ta_2).

6. Show that the midpoint of the line segment joining the points (a, b) and (c, d) is $((a + c)/2, (b + d)/2)$.

7. Prove that the diagonals of a parallelogram bisect each other.

1.2 VECTOR SPACES

In Section 1.1, we saw that with the natural definitions of vector addition and scalar multiplication, the vectors in a plane satisfy the eight properties listed on page 3. Many other familiar algebraic systems also permit definitions of addition and scalar multiplication that satisfy the same eight properties. In this section, we introduce some of these systems, but first we formally define this type of algebraic structure.

Definitions. *A* ***vector space*** *(or* ***linear space****)* V *over a field*[2] F *consists of a set on which two operations (called* ***addition*** *and* ***scalar multiplication****, respectively) are defined so that for each pair of elements* x, y,

[2]Fields are discussed in Appendix C.

in V *there is a unique element* $x + y$ *in* V*, and for each element* a *in* F *and each element* x *in* V *there is a unique element* ax *in* V*, such that the following conditions hold.*

(VS 1) *For all* x, y *in* V, $x + y = y + x$ *(commutativity of addition).*

(VS 2) *For all* x, y, z *in* V, $(x + y) + z = x + (y + z)$ *(associativity of addition).*

(VS 3) *There exists an element in* V *denoted by* $\mathit{0}$ *such that* $x + \mathit{0} = x$ *for each* x *in* V.

(VS 4) *For each element* x *in* V *there exists an element* y *in* V *such that* $x + y = \mathit{0}$.

(VS 5) *For each element* x *in* V, $1x = x$.

(VS 6) *For each pair of elements* a, b *in* F *and each element* x *in* V, $(ab)x = a(bx)$.

(VS 7) *For each element* a *in* F *and each pair of elements* x, y *in* V, $a(x + y) = ax + ay$.

(VS 8) *For each pair of elements* a, b *in* F *and each element* x *in* V, $(a + b)x = ax + bx$.

The elements $x + y$ *and* ax *are called the* ***sum*** *of* x *and* y *and the* ***product*** *of* a *and* x*, respectively.*

The elements of the field F are called **scalars** and the elements of the vector space V are called **vectors**. *The reader should not confuse this use of the word "vector" with the physical entity discussed in Section 1.1: the word "vector" is now being used to describe any element of a vector space.*

A vector space is frequently discussed in the text without explicitly mentioning its field of scalars. The reader is cautioned to remember, however, that *every vector space is regarded as a vector space over a given field, which is denoted by* F. Occasionally we restrict our attention to the fields of real and complex numbers, which are denoted R and C, respectively.

Observe that (VS 2) permits us to unambiguously define the addition of any finite number of vectors (without the use of parentheses).

In the remainder of this section we introduce several important examples of vector spaces that are studied throughout this text. Observe that in describing a vector space, it is necessary to specify not only the vectors but also the operations of addition and scalar multiplication.

An object of the form $(a_1, a_2, \ldots, a_n)$, where the entries $a_1, a_2, \ldots, a_n$ are elements of a field F, is called an ***n*-tuple** with entries from F. The elements

$a_1, a_2, \ldots, a_n$ are called the **entries** or **components** of the n-tuple. Two n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ with entries from a field F are called **equal** if $a_i = b_i$ for $i = 1, 2, \ldots, n$.

Example 1

The set of all n-tuples with entries from a field F is denoted by F^n. This set is a vector space over F with the operations of coordinatewise addition and scalar multiplication; that is, if $u = (a_1, a_2, \ldots, a_n) \in \mathsf{F}^n$, $v = (b_1, b_2 \ldots, b_n) \in \mathsf{F}^n$, and $c \in F$, then

$$u + v = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \quad \text{and} \quad cu = (ca_1, ca_2, \ldots, ca_n).$$

Thus R^3 is a vector space over R. In this vector space,

$$(3, -2, 0) + (-1, 1, 4) = (2, -1, 4) \quad \text{and} \quad -5(1, -2, 0) = (-5, 10, 0).$$

Similarly, C^2 is a vector space over C. In this vector space,

$$(1 + i, 2) + (2 - 3i, 4i) = (3 - 2i, 2 + 4i) \quad \text{and} \quad i(1 + i, 2) = (-1 + i, 2i).$$

Vectors in F^n may be written as **column vectors**

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

rather than as **row vectors** $(a_1, a_2, \ldots, a_n)$. Since a 1-tuple whose only entry is from F can be regarded as an element of F, we usually write F rather than F^1 for the vector space of 1-tuples with entry from F. ♦

An $m \times n$ **matrix** with entries from a field F is a rectangular array of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

where each entry a_{ij} $(1 \le i \le m,\ 1 \le j \le n)$ is an element of F. We call the entries a_{ij} with $i = j$ the **diagonal entries** of the matrix. The entries $a_{i1}, a_{i2}, \ldots, a_{in}$ compose the ***i*th row** of the matrix, and the entries $a_{1j}, a_{2j}, \ldots, a_{mj}$ compose the ***j*th column** of the matrix. The rows of the preceding matrix are regarded as vectors in F^n, and the columns are regarded as vectors in F^m. The $m \times n$ matrix in which each entry equals zero is called the **zero matrix** and is denoted by O.

In this book, we denote matrices by capital italic letters (e.g., A, B, and C), and we denote the entry of a matrix A that lies in row i and column j by A_{ij}. In addition, if the number of rows and columns of a matrix are equal, the matrix is called **square**.

Two $m \times n$ matrices A and B are called **equal** if all their corresponding entries are equal, that is, if $A_{ij} = B_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Example 2

The set of all $m \times n$ matrices with entries from a field F is a vector space, which we denote by $\mathsf{M}_{m\times n}(F)$, with the following operations of **matrix addition** and **scalar multiplication**: For $A, B \in \mathsf{M}_{m\times n}(F)$ and $c \in F$,

$$(A+B)_{ij} = A_{ij} + B_{ij} \quad \text{and} \quad (cA)_{ij} = cA_{ij}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$. For instance,

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & -3 & 4 \end{pmatrix} + \begin{pmatrix} -5 & -2 & 6 \\ 3 & 4 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -2 & 5 \\ 4 & 1 & 3 \end{pmatrix}$$

and

$$-3\begin{pmatrix} 1 & 0 & -2 \\ -3 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 6 \\ 9 & -6 & -9 \end{pmatrix}$$

in $\mathsf{M}_{2\times 3}(R)$. ♦

Example 3

Let S be any nonempty set and F be any field, and let $\mathcal{F}(S, F)$ denote the set of all functions from S to F. Two functions f and g in $\mathcal{F}(S, F)$ are called **equal** if $f(s) = g(s)$ for each $s \in S$. The set $\mathcal{F}(S, F)$ is a vector space with the operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}(S, F)$ and $c \in F$ by

$$(f+g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = c[f(s)]$$

for each $s \in S$. Note that these are the familiar operations of addition and scalar multiplication for functions used in algebra and calculus. ♦

A **polynomial** with coefficients from a field F is an expression of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and each a_k, called the **coefficient** of x^k, is in F. If $f(x) = \mathit{0}$, that is, if $a_n = a_{n-1} = \cdots = a_0 = 0$, then $f(x)$ is called the **zero polynomial** and, for convenience, its degree is defined to be -1;